Lie groups in quantum mechanics

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2006

Abstract

In this survey, we describe some basic mathematical properties of Lie groups and Lie algebras, together with their fundamental usage in quantum mechanics. We are not confined to symmetry applications for computational physics. In fact, we tend to focus on the footing position of Lie groups and Lie algebras as the structure and dynamics description instruments.

1 Introduction

Lie groups are of great importance for mathematical physics. Distinguished features of Lie groups come from combinations of algebraic, i.e. structural, and analytical, i.e. differential properites. While groups are used for symmetry descriptions by many physicists, the Lie groups are of rather general usage.

Let us assume some general dynamical system with its phase flow described by a function $f_{x_0}(t)$, where continuous time development starts at a system state x_0. We require, naturally, first, to be able to describe the dynamics at a stepwise mode and second, the phase flow to be time reversible

$$\begin{aligned} f_{x_0}(t_1+t_2) &= f_{x_1}(t_2) \text{ with } x_1 = f_{x_0}(t_1) \\ x_0 &= f_{x_1}(-t) \text{ with } x_1 = f_{x_0}(t) \end{aligned}$$

Having fulfilled these conditions, the dynamics evolution is endowed with the Lie group structure. Thus we can see Lie groups are prevalent throughout physics.

You can notice it to be familiar to you if you recall the standard derivation of quantum mechanical wave equation. Let us assume the phase volume neither increase/decrease nor it exchanges phase states between its parts. Having assured this one more condition on the system development, we result with the Schrödinger equation as the phase flow description.

We should note this text is not a self-contained introduction into quantum mechanics. First, we suppose the reader is somewhat familiar with standard concepts like Schroedinger equation and operator mechanics. Second, we do not present many topics, e.g. mixed states with density matrices, quantum measurements and dynamics of open systems at all. Even though the non-linearity and non-locality are interesting issues, we concentrate on the deterministic properties of closed systems based on pure quantum states.

2 Mathematics of Lie Groups

Lie groups are accompanied by Lie algebras. Lie groups can be viewed as a general continuous surfaces with some structural properties. Lie algebras are then their linearizations.

Lie groups are connected with transformations as we have seen above. The transformation properties are the core of the representation theory. It can be viewed as the core of physics itself. Current modern high-energy physics theories make usage of representation theory based on category theory. It is beyond the scope of this text.

2.1 Lie structures

We informally introduce manifolds and groups. A *manifold* is a topological space which is locally homeomorphical to an n-dimensional open ball. A *group* is a structure which is endowed with a binary operation $*$ called multiplication satisfying next conditions. The operation is associative, there is a neutral element e, and for any element a of the structure there is an inverse element a^{-1}. It is, for all a, b

$$\begin{aligned} a * (b * c) &= (a * b) * c \\ a * e &= e * a = a \\ a * a^{-1} &= a^{-1} * a = e \end{aligned}$$

Inversion itself can be taken as a unary operation, and the neutral element as a nullary operation.

Lie group is a manifold with group structure where the group operations are analytic mappings.

We briefly summarize the properties:

- neutral element, inverse elements, associativity
- multiplication and inversion smooth operations

Various kinds of invertible square matrices are standard examples of Lie groups. For example, all the regular $n \times n$ matrices over $\mathbf{R}$ form one Lie group. It is the *general linear group* of rank n, $GL_n(\mathbf{R})$. Matrix multiplication is the binary group operation, matrix inversion gives the inverse elements, and identity matrix I_n is the neutral element.

Let us introduce a binary operation, called *Lie bracket*, over a vector space. Lie bracket satisfies bilinerity, antisymmetry and the Jacobi identity. The Jacobi identity states

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

For example, the famous cross product is a Lie bracket for the real space $\mathbf{R^3}$.

Lie algebra is a vector space over a field where the vector space is endowed with a Lie bracket operation.

Lie algebras relate closely to Lie groups. Lie algebras can be made as derivatives of Lie groups at their neutral elements. It was the historical origin of Lie algebras. Frequently, elements of a Lie algebra are called infinitesimal generators.

Let us show some simple examples to become more familiar with the topic. Rather concrete example for Lie groups is $SO(2)$, rotations in two dimensional space $\mathbf{R}^2$. The chosen matrices are orthogonal with their determinants equal to 1. They look like

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

satisfying the $a^2 + b^2 = 1$ constraint.

Now, let us be a bit more general in our study and let us take the set of all unitary $n \times n$ matrices over $\mathbf{C}$, $U(n)$. Unitary matrices are those ones for which we have $A * A^\dagger = A^\dagger * A = I_n$, where I_n is the identity, and $A^\dagger$ is the Hermitian adjoint, i.e. the conjugate transpose.

Matrix product preserves unitarity, thus the set is a group. Both mulitplication and inversion are smooth operations, for the manifold over real numbers. As a result we have the set being a real Lie group.

Then consider one-parameter groups on the $U(n)$. They are differential group homomorphisms from the real line into the $U(n)$. The real line is taken as the additive group. Having chosen one such one-parameter group φ, we have $\varphi(0) = I_n$.

Finally, make derivative along the φ at the neutral element. Since we deal with unitary groups we know the equality below, and we can make derivatives on it. Let us label a given one-parameter group on unitary matrices as $A(t)$, and $A'(t)$ as $B(t)$.

$$\begin{array}{rcl} A(t) * A^\dagger(t) & = & I_n \\ B(t) * A^\dagger(t) & = & -A(t) * B^\dagger(t) \end{array}$$

We made the derivative at the neutral element, i.e. at I_n, thus we have

$$B(1) = -B^\dagger(1)$$

It means the derivatives are skew-Hermitian matrices, i.e. hermitian matrices multiplied by i. We should not be surprised that the resulting matrices form a Lie algebra. In fact, each element of the Lie algebra defines one such one-parameter group. And in the case of matrix structures, it is done by exponentiation.

2.2 Group representations

Lie groups themselves have somewhat complicated topology. It is easier to study just some of their properties. And group representations are for such transfers of depicted group properties into a selected simpler algebraic structure. Commonly, the structures of choice are vector spaces together with their automorphisms. It is so for linear algebra being generally understood.

Together with that, we have principal limitations about what we can grasp by experiments. The potentially available information is embedded into a vector space of states and its automorphisms.

Finally, possible symmetries of studied systems are described by some groups. And influence of the symmetries on differential equations and their solutions is expressed by group representations as well.

Now, let us define group action and its specific case, group representation. *Group action* from a group G on a set S is a group homomorphism from the group G into the set of all permutations of the set S. If we deal with a more structured category than the Set category, we are confined on the available mappings. It means that group action is generally a homomorphism into the set of all the bijective morphisms of the set S. The bijective morphisms are permutations in the case of sets, i.e. the Set category. They are automorphisms in the case of groups, i.e. Grp category.

Representation of a group G on a vector space V is the group action of G on V. It is a group homomorphism from G into automorphisms of V.

Take the set of all the automorphisms of a vector space V of dimension n, i.e. general linear group of degree n, labeled GL_n. Its elements are all the invertible $n \times n$ matrices. It is itself a Lie group, and its Lie algebra consists of all the $n \times n$ matrices with commutator, $g * h - h * g$, being the Lie bracket. We work purely with Lie structures if the represented group is a Lie group too. It is a common case.

It is an adjoint representation what we usually deal with in the field of quantum physics. In such a case, the Lie algebra of the represented Lie group serves as the vector space to be acted on. The action is done naturally by conjugation. Thus, we have our studied Lie group on the input side of the representation homomorphism. And the group of all automorphisms of its Lie algebra is on the output side of the respective representation homomorphism.

A small example can clarify our insight into the field. Take the $SU(2)$ group.

Its elements look like

$$\begin{pmatrix} a+id & c+ib \\ -c+ib & a-id \end{pmatrix}$$

satisfying the $a^2+b^2+c^2+d^2=1$ constraint.

The $su(2)$ Lie algebra is made of traceless skew-Hermitian matrices. They look like

$$\begin{pmatrix} id & c+ib \\ -c+ib & -id \end{pmatrix}$$

with arbitrary b, c, d. It makes $su(2)$ isomorphic with $\mathbf{R}^3$ and $so(3)$. The Lie groups $SU(2)$ and $SO(3)$ are only locally isomorphic.

The adjoint representation of $SU(2)$ is a mapping

$$Ad : g \rightarrow g * h * g^{-1}$$

where $g \in SU(2)$, $h \in su(2)$, and $g * \circ * g^{-1}$ are the conjugate automorphisms of $su(2)$ for each $g \in SU(2)$.

Generally, the domain and the range of a representation mapping can be various structures. An abstract theory dealing with it is category theory. A category consists of objects and morphisms between them. They must obey some rules. We do not go into any details about that. In fact, it is sufficient to take it just intuitively for basic grasping.

For example, the Set category consists of all sets being the objects, and of all mappings being the morphisms. The Grp category have groups as objects and group homomorphisms as the morphisms.

Representations are then done by so called functors. They are mappings between particular categories preserving identity morphisms and morphism compositions. It is an analogy to standard definitions of homomorphisms.

3 Lie groups for Hilbert spaces

Quantum mechanics has simple background of a continuous time evolution. It is determined by unitary time propagators being Lie group operators. Philosophical difficulties come from non-locality of destructive measurements. They are described by Lie operators as well, for now they are Hermitian ones. Computational difficulties come from inherent complexity of particular structure and time development solutions. It can be simplified if some symmetry is contained in studied systems. It is dealt with by usage of Lie groups again.

3.1 System propagators

The playgrounds for quantum mechanics are Hilbert spaces. What we generally need to have is a vector space with some enumeration instruments. Vectors are for system states, enumerations are for state relations determination. Thus, the system is a vector space with inner product. We need complex Hilbert space for quantum mechanics.

A Hilbert space can be both finite and infinite dimensional. Even the simplest systems which have space coordinates, are the infinite dimensional ones. To be more concrete, they are countable dimensional for quantum mechanics cases. It keeps them tractable. Contrary to it, Hilbert spaces of quantum field theory are not separable. They have uncountable large bases.

Nevertheless, we can deal with rather simple ones as well. The two-dimensional complex Hilbert space $\mathbf{C}^2$ is a nice example of a quantum mechanical system. It is a model of a 1/2-spin particle with neglecting its space coordinates.

Dynamics of a closed quantum mechanical systems is described by a unitary propagator:

$$|\psi(t_1)\rangle = \hat{U}(t_1, t_0)|\psi(t_0)\rangle$$

where ψ is a wave function describing the system state, and $\hat{U}(t_1, t_0)$ is the unitary evolution operator of the state from time t_0 to time t_1.

The unitary operator condition comes from next requirements. The causality axiom of quantum mechanics ensures the existence of an operator. The superposition principle makes the operator to be a linear one. Time additivity results in group properties, i.e. in associativity and in having the inverse elements. And finally, the requirement on constant norms leads to the unitarity of the evolution operators.

The $\hat{U}(\circ, \circ)$ operators are linear ones, i.e. they can be realized as matrices with their ranks corresponding to dimensions of the Hilbert space they act on. It is straightforward to see that all such operators on a fixed Hilbert space form a Lie group.

We can look what its Lie algebra looks like. We already know it is formed by skew-Hermitian operators of the same rank. They can be written as Hermitian matrices times i times a real unit dimension and normalization constant. When we choose the constant to be $\hbar$, the Planck's constant, the left Hermitian operators are the Hamiltonians, $\hat{H}$.

Since the operators are matrix ones, the road from Lie algebras back to Lie groups is by exponentiation. Thus, the evolution operators can be written with the help of Hamiltonians as

$$\hat{U}(t_1, t_0) = e^{-i(t_1 - t_0)\hat{H}/\hbar}$$

where time t is the parameter of one-parameter groups used for making the derivatives that construct Lie algebras out of Lie groups. We suppose silently that the systems do not change their charcteristics with time, i.e. that we deal with time-independent system Hamiltonians.

The two presented classes of operators, i.e. propagators and Hamiltonians, are not the only important unitary and Hermitian operators in quantum mechanics.

Measurements are the arbiters elegantiae. Thus we need a tool to connect an abstract Hilbert space to measured quantities. All the observables are realized as Hermitian operators. Besides dealing with time and energy, one usually measures positions and momenta. We study them in the sequel.

3.2 Commutation relations

For now, we concern ourselves in non-commutativity properties of quantum mechanics. Commutators $[A, B] = A * B - B * A$ are the quantum analogy of classical Poisson brackets

$$\{a, b\} = \sum \left(\frac{\partial a}{\partial q^i} \frac{\partial b}{\partial p_i} - \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial q^i} \right)$$

In fact, both of them can act as the Lie bracket for the respective Lie algebra.

Everybody knows there are commutation relations between the positions and momenta. And now comes one peculiarity. One uses the term of representation both for projections of abstract states, i.e. vectors of a Hilbert space, on vector spaces of observables, and for realization of the Heisenberg commutation relations. In turn, it reveals both notions express the same mathematical process: choise of a concrete system to work with.

Projections are mappings from hard-to-grasp abstract spaces into familiar spaces of vectors for states with matrices for operators, even though they are usually infinite dimensional. These projections are smooth homomorphisms from abstract into concrete Hilbert spaces. Bases of the new, i.e. concrete, Hilbert spaces are spanned by eigenvectors of a chosen complete system of observables, i.e. of Hermitian operators.

Most of the other observables are just reincarnations of the coordinate $\hat{x}$ - momentum $\hat{p}$ pair. With one notable exception: spins. However, even the spins can be viewed as an outcome of a fulfilment of some commutation relations, on angular momenta in this case.

From one point of view, alpha and omega of quantum theory is contained in commutation relations. They are

$$[\hat{Q}_i, \hat{Q}_j] = 0,\ [\hat{P}_i, \hat{P}_j] = 0,\ [\hat{Q}_i, \hat{P}_j] = i\hbar\delta_{i,j}$$

for canonically conjugate operators $\hat{Q}$ and $\hat{P}$. The canonical form of them is $[\hat{x}, \hat{p}] = i\hbar\hat{1}$. The commutation relations can be used to construct a Lie algebra, the Heisenberg algebra, with its Lie bracket being the commutation relations where $i\hbar$ is replaced by a general element C that commutes with all the elements.

Heisenberg algebra h_n is a $2n+1$ dimensional real Lie algebra. Its basis consists of $P_1, \ldots, P_n$, $Q_1, \ldots, Q_n$, C. Its Lie bracket is defined as follows:

$$[Q_i, Q_j] = 0,\ [P_i, P_j] = 0,\ [C, \circ] = 0,\ [Q_i, P_j] = C\delta_{i,j}$$

Nontrivial representations of a given finite-dimensional Heisenberg algebra on a respective separable Hilbert space are unitarily equivalent. To be fair, we should add two conditions. First, we have to choose the constant C, i.e. the right Planck's constant. Second, the representations have to be regular in the sense we can make their Lie groups by exponentiation. All the usual representations obey that. It was a stepping stone for prooving the equivalence of Schroedinger's wave mechanics and Heisenberg's matrix mechanics.

The unitary equivalence means for each pair of representations there exists a unitary operator transforming the respective states, i.e. vectors, and respective observables, i.e. Hermitian operators, into each other. Having two representations, I and II, and a Hermitian observable A, a suitable unitary operator U is used as follows:

$$A_I = U^\dagger A_{II} U,\ A_{II} = U A_I U^\dagger$$

While the Hermitian property of observables ensures real eigenvalues and a suitable basis, the unitary equivalence is necessary for having measurement results invariant with respect to chosen representations. It depicts another cooperation between unitary and Hermitian operators in quantum mechanics. Quantum field theories do not bear such a nice property of unitary equivalence, since their Hilbert spaces are not separable. It results in e.g. unsuitability of Fock's spaces then.

We have dealt with representations of Lie algebras above. It could be done with Lie groups as well, due to the ability to make the exponentiation. All the categories we have dealt with, can be thought as the abstract categories. The category of commutation relations together with some chosen faithfull functors can be thought as a concrete category then. The functors work as the unitarily equivalent representations.

3.3 Dynamics invariances

We can see two classes of invariances of solutions of dynamic systems. First, there are transformations to which we generally require our solutions to be invariant. Second, we can face special situations where some extra symmetry provides us with another invariance of a respective solution.

The former type of invariances can state e.g. possibility to make a measurement in an arbitrary time. Experiment results should not depend on specific setting of zero time. Generally, it is about covariance with changes of frames, usually Galilean or Lorentz ones.

The latter type deals with ad-hoc symmetries. For example, spherically symmetrical potential endows solutions of its systems with invariances with respect to changes of angles. Knowledge of it can be helpful with resolving otherwise complex systems.

According to the Noether's theorem, each of the differentiable invariances provides us with a conservation law. Again, the invariance-conservation pairs are those pairs of commutation relations. In the case of time-energy pair we need more thoughts, of course.

A sketch of proof for a time-independent symmetry. We write $L = L(q, \dot{q})$ for Lagrangian, c for the invariance quantity and C for the alleged conserved quantity.

$$\begin{aligned} C &= p * \partial q(c)/\partial c \\ \dot{C} &= \dot{p} * \partial q(c)/\partial c + p * \partial \dot{q}(c)/\partial c \end{aligned}$$

Now, we have $p = \partial L/\partial \dot{q}$ by definition, and we know $\dot{p} = \partial L/\partial q$ from the equation of motion.

$$\begin{aligned} \dot{C} &= \partial L/\partial q * \partial q(c)/\partial c + \partial L/\partial \dot{q} * \partial \dot{q}(c)/\partial c \\ \dot{C} &= d/dc\ L(q(c), \dot{q}(c)) \end{aligned}$$

Finally, according to the assumption of the invariance $dL/dc = 0$, we have $\dot{C} = 0$ as a result.

These properties are important from the dynamical point of view. They can be taken as a dynamics counterpart to the structural description of quantum mechanics around commutation relations. In fact, path integral formulation of quantum mechanics that is based on the exploitation of Lagrangians, is a rather fruitful concept. It even provides the all-paths-explanation for classical mechanics where systems have to choose their single paths in advance.

The functional character of variables is prevalent throughout quantum theory. For example, canonically conjugate operators are gained by quantization of functionally conjugated variables. It means we deal with quantized canonical momenta $p = \partial L/\partial \dot{q}$ instead of the mechanical momenta $m\dot{q}$. While the two notions coincide in many situations, they differ e.g. under presence of magnetism.

One more example on functional features of quantum theory is the regularization by the analytic continuation of functions which converge on some open subsets of the complex plane. Then, we forget the mechanical definition of the function, and we deal with it according to its functional properties.

In the current section, quantum mechanics concepts were explored with respect to their mutual connections of algebraic and analytic properties. This structure - dynamics duality provides us with a firm base for quantum mechanics. We have tried to present the topics in the language of Lie groups and algebras since they are the mainstream gears for studying the combination of algebraic and analytic notions.

4 Spin characteristics

Standard classification of particles onto fermions and bosons is done according to their spins. As quantum mechanics works mainly on electrons, 1/2-spin is vastly studied. We describe this case as well. First, we start with the spin by commutation relations. Then we produce it as an adjoint representation.

4.1 Spin by commutations

Taking the classical definition of angular momentum $\vec{L} = \vec{r} \times \vec{p}$ together with quantum commutation relations for $\hat{x}$, $\hat{p}$, we result with commutation relations for the quantum angular momentum:

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}L_k, \;\; [\hat{L}^2, \hat{L}_i] = 0$$

where $\hat{L}_i$ are angular momentum components and $\hat{L}^2$ is defined as $\sum \hat{L}_i^2$. Assuming spherically symmetric system, we choose $\hat{L}^2$ and $\hat{L}_z^2$ operators to deal with. We take $|\lambda, \mu\rangle$ as the eigenstates common to both $\hat{L}^2$ with $\lambda\hbar^2$ eigenvalues, and $\hat{L}_z$ with $\mu\hbar$ eigenvalues.

Next to it, we define annihilation and creation operators in the standard way

$$\begin{aligned} \hat{L}_- &= \hat{L}_x - i\hat{L}_y \\ \hat{L}_+ &= \hat{L}_x + i\hat{L}_y \end{aligned}$$

Their actions on $|\lambda, \mu\rangle$ states can be easily investigated with usage of the above commutation relations

$$\begin{aligned} \hat{L}^2(\hat{L}_\pm|\lambda, \mu\rangle) &= \hat{L}_\pm(\hat{L}^2|\lambda, \mu\rangle) = l\hbar^2\hat{L}_\pm|\lambda, \mu\rangle \\ \hat{L}_z(\hat{L}_\pm|\lambda, \mu\rangle) &= (\pm\hbar\hat{L}_\pm + \hat{L}_\pm\hat{L}_z)|\lambda, \mu\rangle = (\mu \pm 1)\hbar\hat{L}_\pm|\lambda, \mu\rangle \end{aligned}$$

Since component values are bounded by the square values, we should have some maximal and minimal eigenvalues of $\hat{L}_z$ with respective eigenstates for belonging to a $\hat{L}^2$ eigenstate. And from

$$\langle\lambda, \mu| \; \hat{L}^2 - \hat{L}_z^2 \; |\lambda, \mu\rangle \;=\; \langle\hat{L}_x \; \lambda, \mu \mid \hat{L}_x \; \lambda, \mu\rangle + \langle\hat{L}_y \; \lambda, \mu \mid \hat{L}_y \; \lambda, \mu\rangle \geq 0$$

we have $\lambda\hbar^2 \geq \mu^2\hbar^2$, i.e. $\sqrt{\lambda} \geq |\mu|$. It leads to a maximal and a minimal value of μ for a fixed λ. We write $m_{max}\hbar$, $m_{min}\hbar$ for the maximal, minimal $\mu\hbar$ eigenvalues, respectively.

Actually possible eigenvalues can be gained from vanishing requirements on eigenstates of the maximal and minimal eigenvalues:

$$\begin{aligned} \hat{L}_+|\lambda, m_{max}\rangle &= 0|\rangle \\ \hat{L}_-\hat{L}_+|\lambda, m_{max}\rangle &= (\lambda - m_{max}^2 - m_{max})\hbar^2|\lambda, m_{max}\rangle = 0|\rangle \end{aligned}$$

and in a similar way for the minimal $\mu\hbar$ eigenvalue, i.e. for $m_{min}\hbar$. As a result, we have two quadratic equations with solutions

$$\lambda = m_{max}(m_{max} + 1) = m_{min}(m_{min} - 1)$$

with only $m_{max} = -m_{min}$ reasonable equality, i.e. with $m_{max} \geq m_{min}$. The m_{max} is usually labeled as l. We write it so in the next too.

Thus, we have the $-l\hbar, ..., l\hbar$ possible $\mu\hbar$ eigenvalues for a $\lambda\hbar^2 = l\hbar^2$ given eigenvalue. The sequence of possible $\mu\hbar$ eigenvalues increases by $\hbar$-sized steps, as it was given by the creation and annihilation operators. Thus, we have integer amount of the values and it results on l being either integer or half-integer.

The cases with integer l – and an even amount of possible $\mu\hbar$ eigenvalues, are the quantum analogy for the classical angular momentum. The cases with half-integer l – and an odd amount of possible $\mu\hbar$ eigenvalues, are an intrinsic quantum property. It comes as a pure fulfilment of the commutation relations.

4.2 Spin by representations

Dirac's solution of a relativistic version of the wave equation leads into 1/2-spin as well. The solution is written as a spinor, i.e. a vector-like object changing its sign under 2π rotation.

Both ladder, i.e. creation and annihilation, operators and spinors are gained by rewriting some terms, sums of squares, into multiplications. Ladder operators can be gained by rewriting the $(a^2 + b^2)$ term into the $(a + ib)(a - ib)$ term. Spinors for the 1/2-spin case can be gained by rewriting the $\sum_i^3 a_i^2$ term into the $(\sum_i^3 a_i)^2$ term. While the above ladder operators are a realization of commutator conditions $[\alpha_m, \alpha_n^\dagger] = \delta_{m,n}I$, the presented spinors are a realization for anticommutator conditions $\{\alpha_i, \alpha_j\} = 2\delta_{i,j}I$.

In the next, we confine ourselves to the case of 1/2-spin spinors. We gain such spinors as a representation of $SU(2)$ group that is itself a double-cover of $SO(3)$ group. Both of them are Lie groups. First, we present basic ideas on their topologies.

The $SU(2)$ group can be viewed as the group of rotations in the $\mathbf{C}^2$ space. The $SO(3)$ group is for rotations in the $\mathbf{R}^3$ space. Rotations in the real 3-dimensional space are in $1-1$ correspondence with vectors determined by axis lines together with rotation angles. Thus the $SO(3)$ group can be viewed as a ball of π radius with identifying the opposite points on its surface.

We have already presented the $SU(2)$ group. It is isomorphic to the S^3 sphere, i.e. surface of the unit radius ball in the R^4 space. Since S^3 is contractible, i.e. each closed path can be smoothly contracted to a point, $SU(2)$ has the property as well.

The $SO(3)$ group is not contractible. Nevertheless, we have $2-1$ homomorphism from $SU(2)$ onto $SO(3)$. It is a local isomorphism and it transfers the contraction property. It identifies opposite points on the S^3 sphere. For example, a full path over S^3 is imaged as a 4π rotation. Thus, a 4π rotation in R^3 is contractible, contrary to a 2π rotation.

The contractibility has some topological implications. Take a solid object and anchore it by strings in its respective space. The strings get intertwined during a rotation. However, they can be untwined if the rotation path - assuming a closed path - can be contracted to a point.

Now, just make the adjoint representation of $SU(2)$ on its $su(2)$ algebra. It assigns a $g{*}h{*}g^{-1}$ to each h element of $su(2)$ and g matrix of $SU(2)$. Remember that $su(2)$ is isomorphic to $so(3)$ and thus to R^3. Thus the action of $SU(2)$ can be displayed as an action on vectors of R^3, or on angular velocities in R^3.

Not only we have the $su(2)$ to $so(3)$ isomorphism, we have the $2-1$ homomorphism from $SU(2)$ onto $SO(3)$ too. Thus the acting matrices can be displayed as rotations in R^3. However, each point of $SO(3)$ has its preimage comprised of two respective opposite points of $SU(2)$.

Draw a path of a full rotation of an element of the adjoint representation image, i.e. of $su(2)$, $so(3)$ or R^3. It uses consequently two opposite elements of

$SU(2)$ with the same image on $SO(3)$ by the $2-1$ homomorphism. Thus, the first pass of the given 2π rotation element of $SO(3)$ is done while the respective rotated object is just half-rotated, i.e. while it is multiplied by -1. The rotated object returns into itself only during the second pass of the 2π rotation of $SO(3)$. It is the 4π rotation that is required for a spinor to be rotated into itself.

Finally, we take a look on spin-statistics connections. We take the fermion case. We know the -1 multiplication of a wave function under a 2π rotation is caused by the action of $SU(2)$. The second issue on a fermion wave function is that it is multiplied by -1 when two fermions are interchanged. It is in a direct connection with the antisymmetric form of the respective wave function.

Tangloids are a model to put the two properties together. A tangloid is formed by two solid pieces connected together by several strings. Usually, three strings are used, and it is the least necessary amount of the strings. Both the solid pieces and strings are just classical objects of the ordinary world.

First, the strings serve as anchors that can be untwined after a 4π rotation of one of the tangloid pieces. Remember, paths of 4π rotations are contractible, contrary to paths of 2π rotations.

Second, we can see that an interchange of the two tangloid pieces is equivalent to a 2π rotation of one of them. Just make the interchange in several steps. First, put the tangloid pieces together – without any rotation. Then, rotate the tangloid at whole by π – without any string intertwining. After that, put the tangloid pieces into their final positions. Strings are still untwined. Both tangloid pieces are at their new positions, however their are both π rotated. The final step is to rotate both of them by π at the same direction.

It is questionable whether the tangloid analogy is a result of the actual physical reality or whether it is just by an accident. There are some axiomatizations of quantum field theory, e.g. the Wightman one, which have spin-statistics connections as a proovable theorem. According to some physicists, the current axiomatizations of quantum field theory are the right way, with tangloids and other similar structures being a vapour. Contrary to that, some other physicists suppose the tangloids are more fundamental. Then, the above mentioned axiomatizations are supposed to be constructed with prior spin-statistics connections embedding.

We have dealt with quantum mechanics that has one rather weak point. It takes systems as having fixed point of particles. It is unsufficient for e.g. light absorption and emission. Physical interactions themselves are based on virtual particles action at all. Nevertheless, Lie structures are still helpful.

Lie group representations are used as a powerful technics throughout quantum field theory. We should mention, the right representations are supposed to be the irreducible ones. They are those which have no nontrivial invariant subspaces. For example, $SO(1,3)$ group is used for the Lorentz covariance. Other groups, like the isospin group, are used for constructing the standard model of particles.

5 Web resources

One can find many resources about Lie structures all around the web. Two rather general sites are 'Stanford Encyclopedia of Philosophy' and 'Wikipedia, the free encyclopedia'. While the former has usually more complete covering of a topic, the latter contains much more articles with many links to other resources.

- Stanford Encyclopedia of Philosophy
  `http://plato.stanford.edu/`
- Wikipedia, the free encyclopedia
  `http://en.wikipedia.org/`

Besides that, many mathematicians and physicist have insightful material available on the web, among them notably John Baez and Peter Woit.

- John Baez `http://math.ucr.edu/home/baez/`
- Peter Woit `http://www.math.columbia.edu/~woit/`

Notable texts are lectures on 'Lie Groups and Quantum Mechanics' by Michael Weiss, 'Lie Groups and Representations' by Peter Woit, and an introduction to connections between quantum theory and category theory 'Categories, Quantization, and Much More' by John Baez.

- Weiss, M., "Lie groups and Quantum Mechanics"
  `http://math.ucr.edu/home/baez/lie/lie.html`
- Woit, P., "Lie Groups and Reresentations"
  `http://www.math.columbia.edu/~woit/repthy.html`
- Baez. J., "Categories, Quantization, and Much More"
  `http://math.ucr.edu/home/baez/categories.html`

Interesting discussions not only about quantum mechanics are on usenet group Sci.Physics.Research that have web archive.

- Sci.Physics.Research newsgroup
  `http://www.lns.cornell.edu/spr/`

For those who are interested in computational applications on abstract topics, GAP is a software for symbolic manipulations centered around group theory. It contains a Lie algebra package.

- GAP – Groups, Algorithms, Programming
  `http://www.gap-system.org/`